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Superinflation and Quintessence in the Presence of an Extra Field

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We discuss the implication of the introduction of an extra field to the dynamics of a scalar field conformally coupled to gravitation in a homogeneous isotropic spatially flat universe. We show that for some reasonable parameter values the dynamical effects are similar to those of our previous model with a single scalar field. Nevertheless, for other parameter values new dynamical effects are obtained.

KEY WORDS: scalar field; nonminimal coupling; fixed points.

1. INTRODUCTION

Recently, the authors investigated (Gunzig *et al.*, 2000, 2001) the dynamics of a spatially flat universe dominated by a self-interacting conformally coupled scalar field according to the action

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(-\frac{R}{\kappa} + g^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi - 2V(\psi) + \xi R\psi^2\right), \qquad (1)$$

where R denotes the scalar curvature, ψ is the scalar fields, $\kappa \equiv 8\pi G$ (G being Newton's constant). A cosmological constant, if present, is incorporated in the scalar field potential

$$V(\psi) = \frac{3\alpha}{\kappa}\psi^2 - \frac{\Omega}{4}\psi^4 - \frac{9\omega}{\kappa^2}. \qquad (2)$$

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Many new and interesting dynamical phenomena were discovered and its cosmological interpretation were discussed. Special attention was given to the conformal coupling case with $\xi = 1/6$. This choice for the coupling constant is motivated by physical arguments from particle theories (Faraoni, 1996, 2000; Sonego and Faraoni, 1993) and from scale invariance at the classical level (Callan *et al.*, 1970). The nonminimal coupling is also required by first loop corrections (Birrell and Davies, 1980; Ford and Toms, 1982; Nelson and Parangaden, 1982; Parker and Toms, 1985). One interesting effect discussed in Gunzig *et al.* (2000, 2001) is superinflation characterized by $\dot{H} > 0$, and that can only be achieved by a nonminimal coupling. Here we consider the robustness of these dynamical phenomena with respect to the introduction into the model of a second massless scalar field. We also discuss some new types of asymptotic solutions arising from the introduction of the extra field.

The structure of the paper is the following: in Section 2 we present the model and some definitions. The fixed points and their stability are given in Section 3. Additional asymptotic solutions are discussed in Section 4 and the paper is closed with some concluding remarks in Section 5.

2. THE MODEL

We consider the conformally coupled theory described by the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(-\frac{R}{\kappa} + g^{\mu\nu}\partial_\mu\psi_1\partial_\nu\psi_1 - 2V_1(\psi_1)\right.$$
$$\left. + \frac{1}{6}R\psi_1^2 + g^{\mu\nu}\partial_\mu\psi_2\partial_\nu\psi_2 - 2V_2(\psi_2) + \frac{1}{6}R\psi_2^2 + \beta\psi_1^2\psi_2^2\right). \qquad (3)$$

We use the full conserved scalar field stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\psi_1\partial_\nu\psi_1 - \frac{1}{6}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\left(\psi_1^2\right) + \frac{1}{6}G_{\mu\nu}\psi_1^2$$
$$- \frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\psi_1\partial^\alpha\psi_1 - 2V_1(\psi_1)\right)$$
$$+ \partial_\mu\psi_2\partial_\nu\psi_2 - \frac{1}{6}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\left(\psi_2^2\right) + \frac{1}{6}G_{\mu\nu}\psi_2^2$$
$$- \frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\psi_2\partial^\alpha\psi_2 - 2V_2(\psi_2)\right) - \frac{\beta}{2}\psi_1^1\psi_2^2 \qquad (4)$$

(where $G_{\mu\nu}$ is the Einstein tensor), thereby avoiding the use of any effective coupling constant in the Einstein equations. Moreover, this consideration of the energy–momentum tensor $T_{\mu\nu}$ together with an adequate self-consistent treatment of Einstein's and Klein–Gordon equations eliminates the artificial pathologies associated, in Einstein's equations, with the "critical factor" $1 - (\psi_1^2 + \psi_2^2)\kappa/6$. The consensual attitude encountered in the literatures is indeed that the dynamics is ill-defined when this factor is negative or vanishes. That this is not the case follows from the fact that the Klein-Gordon equations, when properly combined with Einstein's equations, completely destroy this factor and its dynamical consequences. This property holds for the homogeneous as well as inhomogeneous cases. The important facts will be presented in a forthcoming publication.

Let us consider the following form for the potentials V_1 and V_2:

$$V_i(\psi_i) = \frac{3\alpha_i}{\kappa}\psi_i^2 - \frac{\Omega_i}{4}\psi_i^4 - \frac{9\omega_i}{\kappa^2}. \tag{5}$$

The parameter α_i is related to the mass of the particle by $m_i = \sqrt{6\alpha_i/\kappa}$. As in the first papers, we are interested in the dynamics of a spatially flat Friedmann–Robertson–Walker universe with line element $ds^2 = d\tau^2 - a^2(\tau)(dx^2 + dy^2 + dz^2)$. Also we will restrict the potential in (5) to the case $\alpha_1 \equiv \alpha$, $\alpha_2 = 0$, $\Omega_1 \equiv \Omega$, $\Omega_2 = 0$, $\omega_1 \equiv \omega$, and $\omega_2 = 0$. This corresponds to a massive field coupled to a massless field. More generic cases will be considered in a forthcoming publication. The energy density and pressure associated to the scalar fields are

$$\sigma = \sigma_1 + \sigma_2 - \frac{\beta}{2}\psi_1^2\psi_2^2, \tag{6}$$

$$p = p_1 + p_2 + \frac{\beta}{2}\psi_1^2\psi_2^2, \tag{7}$$

where

$$\sigma_i = \frac{\dot{\psi}_i^2}{2} + \frac{1}{2}H^2\psi_i^2 + \frac{1}{2}H\partial_\tau\left(\psi_i^2\right) + V_i(\psi_i), \tag{8}$$

$$p_i = \frac{\dot{\psi}_i^2}{2} - \frac{1}{6}\left[2H\partial_\tau\left(\psi_i^2\right) + \partial_{\tau\tau}^2\left(\psi_i^2\right)\right] - \frac{1}{6}(2\dot{H} + 2H^2)\psi_i^2 - V_i(\psi_i). \tag{9}$$

The action (3) then implies the trace equation

$$R = -6(\dot{H} + 2H^2) = -\kappa(\sigma - 3p), \tag{10}$$

the energy constraint

$$3H^2 = \kappa\sigma = 0, \tag{11}$$

and the Klein-Gordon equations for the two scalar fields,

$$\ddot{\psi}_1 + 3H\dot{\psi}_1 - \frac{1}{6}R\psi_1 + \frac{dV_1}{d\psi_1} + \beta\psi_1\psi_2^2 = 0, \tag{12}$$

$$\ddot{\psi}_2 + 3H\dot{\psi}_2 - \frac{1}{6}R\psi_2 + \frac{dV_2}{d\psi_2} + \beta\psi_1^2\psi_2 = 0. \tag{13}$$

Using the Klein-Gordon equations (12) and (13) and the expressions for σ and p we obtain

$$\sigma - 3p = 4(V_1 + V_2) - \psi_1 \frac{dV_1}{d\psi_1} - \psi_2 \frac{dV_2}{d\psi_2}. \tag{14}$$

This simple form holds for any interaction of the form $\beta\psi_1^\gamma \psi_2^\delta$, provided $\gamma + \delta = 4$. Finally, the energy constraint is

$$3H^2 - \kappa\sigma = 3H^2 - \frac{1}{2}\kappa\dot{\psi}_1^2 - \frac{1}{2}\kappa H^2\psi_1^2 - \kappa H\psi_1\dot{\psi}_1 - \kappa V_1$$
$$- \frac{1}{2}\kappa\dot{\psi}_2^2 - \frac{1}{2}\kappa H^2\psi_2^2 - \kappa H\psi_2\dot{\psi}_2 - \kappa V_2 + \frac{1}{2}\kappa\beta\psi_1^2\psi_2^2 = 0. \tag{15}$$

Now solving Eqs. (14) and (15) for the derivatives $\dot{H}$ and $\dot{\psi}_1$ we obtain

$$\dot{H} = -2H^2 + \frac{2}{3}\kappa V_1 + \frac{2}{3}\kappa V_2 - \frac{1}{6}\kappa\psi_1 \frac{dV_1}{d\psi_1} - \frac{1}{6}\kappa\psi_2 \frac{dV_2}{d\psi_2}, \tag{16}$$

$$\dot{\psi}_1 = \frac{1}{\kappa}(-\kappa H\psi_1 \pm \sqrt{G}), \tag{17}$$

where

$$G \equiv 6\kappa H^2 - \kappa^2\dot{\psi}_2^2 - 2\kappa^2 H\psi_2\dot{\psi}_2 - 2\kappa^2(V_1 + V_2) - \kappa^2\beta\psi_1^2\psi_2^2 - \kappa^2 H^2\psi_2^2. \tag{18}$$

The region $G < 0$ in phase space is physically forbidden.

Using Eq. (17) allows to rewrite system [(6)–(10)] in the following equivalent form:

$$\dot{H} = -2H^2 + \frac{\kappa(\sigma - 3p)}{6}, \tag{19}$$

$$\dot{\psi}_1 = -H\psi_1 \pm \frac{\sqrt{G}}{\kappa}, \tag{20}$$

$$\ddot{\psi}_2 = -3H\dot{\psi}_2 + \frac{1}{6}R\psi_2 - \frac{dV_2}{d\psi_2} - \beta\psi_1^2\psi_2. \tag{21}$$

The system of ODE [(10), (12), (13)] is a five-dimensional system, but owing to energy constraint (11) the real phase space is constrained into a four-dimensional manifold. The situation is similar to the one-field case, where the original system is three dimensional, but the energy constraint enforces the motion to take place on a two-dimensional manifold. In the one-field case, the plane (H, ψ) is divided in to sectors by some straight lines, corresponding to regions of distinct state equations for the fluid ψ. In the present case, the projection of the real phase space onto the (H, ψ_1) plane conserves some of those sectors. For example, from

$$\sigma - 3p = 6\alpha\psi_1^2, \tag{22}$$

valid for any quartic interaction potential between ψ_1 and ψ_2, one can see that the region $\psi_1 = 0$ still corresponds to a state equation of pure radiation.

3. FIXED POINTS

In our case De Sitter solutions correspond to fixed points except for the fixed point at the origin. The fixed points are obtained from the conditions $\dot{\psi}_1 = \dot{\psi}_2 = 0$. The stability of the fixed points is determined from the linearized equations in a neighborhood of each point, except for special cases discussed below. For all the fixed points we have $s_1, s_2, s_3 = \pm 1$. The existence conditions for the fixed points is such that the arguments of the square roots are positive:

- Type A

$$H = s_1\sqrt{\frac{3}{\kappa}}\sqrt{\frac{-\beta}{\alpha+\beta}}, \tag{23}$$

$$\psi_1 = s_2\sqrt{\frac{6}{\kappa}}\sqrt{\frac{\omega}{\alpha+\beta}}, \tag{24}$$

$$\psi_2 = s_3\sqrt{\frac{6}{\kappa}}\sqrt{\frac{\omega(\Omega+\beta)-\alpha(\alpha+\beta)}{\beta(\alpha+\beta)}}. \tag{25}$$

- Type B

$$H = s_1\sqrt{\frac{-3\omega}{\kappa}}, \tag{26}$$

$$\psi_1 = \psi_2 = 0. \tag{27}$$

- Type C

$$H = s_1\sqrt{\frac{3}{\kappa}}\sqrt{\frac{\alpha^2-\omega\Omega}{\Omega-\alpha}}, \tag{28}$$

$$\psi_1 = s_2\sqrt{\frac{6}{\kappa}}\sqrt{\frac{\alpha-\omega}{\Omega-\alpha}}, \tag{29}$$

$$\psi_2 = 0. \tag{30}$$

Fixed points of Type B are on the $G = 0$ surface.

In order to discuss the stability of these fixed points let us consider the generic case given by a system of the form

$$\dot{x}_i = F_i(x_1, \ldots, x_n) \equiv F(x), \quad i = 1, \ldots, n. \tag{31}$$

A fixed point $\overline{x}$ satisfies $F_i(\overline{x}) = 0$. Its stability is usually established by linearizing (31) in a neighborhood of $\overline{x}$. In this way we write $x = \overline{x} + \delta x$ and obtain (31):

$$\frac{d\,\delta x}{dt} = \mathbf{J}\delta x + \mathcal{O}(\delta x^2), \tag{32}$$

where $\mathbf{J}$ is defined by

$$\mathbf{J}_{ij} \equiv \left.\frac{\partial F_i}{\partial x_j}\right|_{x=\overline{x}}, \tag{33}$$

and $\delta x \equiv (\delta x_1, \ldots, \delta x_n)$. If Det($\mathbf{J}$) $\neq 0$ and if the eigenvalues λ_i of $\mathbf{J}$ are such that Re(λ_i) $\neq 0$ then the Hartman–Grobmann theorem ensures that the linearized system obtained by retaining only linear terms in (32) is topologically equivalent to the original system (31) at a neighborhood of the fixed point. The real parts of the eigenvalues then determine the local stability of the fixed points. The stability of the fixed points on the $G = 0$ surface cannot be studied using this method as Det($\mathbf{J}$) $= 0$. To overcome this problem we will consider the system formed by Eqs. (12), (13), and (16). The only points to be considered separately are of Type B and their stability is studied by other methods.

Equations (12) and (13) can be rewritten as a first-order equations by introducing the new variables $\phi_1 \equiv \dot{\psi}_1$ and $\phi_2 \equiv \dot{\psi}_2$. In this way we identify the variables x_i in (31) with $(H, \psi_1, \phi_1, \psi_2, \phi_2)$ and obtain from (33) the following form for the $\mathbf{J}$ matrix computed at the fixed point of coordinates $(\overline{H}, \overline{\psi}_1, \overline{\phi}_1, \overline{\psi}_2, \overline{\phi}_2)$:

$$\mathbf{J} = \begin{pmatrix} -4\overline{H} & 2\alpha\overline{\psi}_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & C_1 & -3\overline{H} & -2\beta\overline{\psi}_1\overline{\psi}_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2(\alpha+\beta)\overline{\psi}_1\overline{\psi}_2 & 0 & C_2 & -3\overline{H} \end{pmatrix}, \tag{34}$$

with

$$C_1 = -\beta\overline{\psi}_2^2 + 3(\Omega - \alpha)\overline{\psi}_1^2 + \frac{6}{\kappa}(\omega - \alpha) \tag{35}$$

and

$$C_2 = -(\alpha + \beta)\overline{\psi}_1^2 + 6\frac{\omega}{\kappa}. \tag{36}$$

The eigenvalues of $\mathbf{J}$ are

$$\lambda = -\frac{3}{2}\overline{H} \pm \frac{1}{2}\sqrt{9\overline{H} + 9A \pm \frac{2}{\kappa}\sqrt{B}}, \tag{37}$$

$$\lambda = -4\bar{H}, \tag{38}$$

where the $\pm$ signs are to be considered independently of each other,

$$A = \frac{2}{9}(3\Omega - \beta - 4\alpha)\bar{\psi}_1^2 - \frac{2}{9}(\beta + 4\alpha)\bar{\psi}_2^2 + \frac{4}{3\kappa}(2\omega - \alpha) \tag{39}$$

and

$$B = \kappa^2\big(\beta^2\bar{\psi}_2^2 + (20\alpha\beta + 14\beta^2 - 6\beta\Omega)\bar{\psi}_1^2\bar{\psi}_2^2 + (4\alpha^2 + \beta^2 - 4\alpha\beta - 12\Omega\alpha + 6\beta\Omega + 9\Omega^2)\bar{\psi}_1^4\big). \tag{40}$$

A fixed point is stable, unstable, or a saddle point if the real parts of the five eigenvalues are all negative, all positive, or have different signs, respectively. Points of Types A and C are saddle points. The stability of fixed points of Type B is discussed in the next section.

4. ASYMPTOTIC DYNAMICS

Two kinds of asymptotic dynamics are more relevant for our model: attractive fixed points and diverging solutions. For the one-field model the fixed point at $H = \psi = 0$ acts as a an attractor and plays an important role in our approach (Gunzig *et al.*, 2000, 2001). The fixed point at

$$\psi_1 = \dot{\psi}_1 = \psi_2 = \dot{\psi}_2 = 0 \quad \text{and} \quad H = \sqrt{-3\omega/\kappa} \tag{41}$$

has a similar role. For this purpose we will consider her the special case of a nonnegative cosmological constant $\omega \leq 0$. This implies that $\dot{H} \geq 0$ for $H = 0$ as $\sigma - 3p$ is positive for ω negative and therefore the region $H > 0$ is invariant under the dynamics. Let us now consider the function

$$L = \frac{1}{2}\dot{\psi}_1^2 + \frac{1}{2}\frac{\beta}{\beta - \alpha}\dot{\psi}_2^2 + \frac{3\alpha}{\kappa}\psi_1^2 + \frac{1}{4}(\alpha - \Omega)\psi_1^4 - \frac{1}{2}\beta\psi_1^2\psi_2^2. \tag{42}$$

Its total time derivative modulo Eqs. (12), (13), and (16) is given by

$$\dot{L} = -3H\left(\dot{\psi}_1^2 + \frac{\beta}{\beta - \alpha}\dot{\psi}_2^2\right) + \frac{6\omega}{\kappa}\psi_1\dot{\psi}_1 + \frac{6\beta\omega}{(\beta - \alpha)\kappa}\psi_2\dot{\psi}_2. \tag{43}$$

For parameter values such that $\omega = 0$, $\beta < 0$, and $\Omega \leq \alpha$, the function L is a Lyapunov function for the fixed point (41) and satisfy the properties

$$L \geq 0 \tag{44}$$

and

$$\dot{L} \leq 0. \tag{45}$$

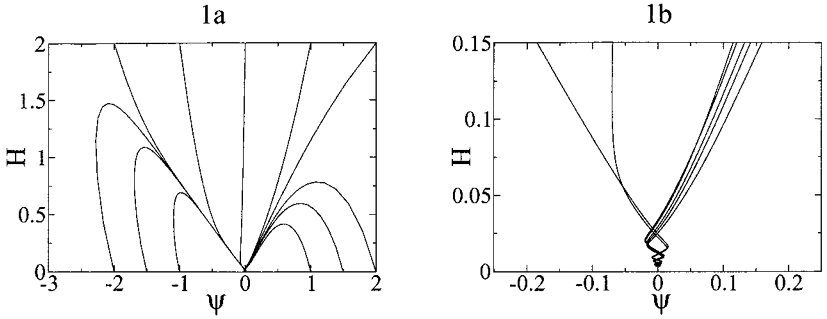

Fig. 1. Example of solutions projected in the (H, ψ_1) plane for $\omega = 0$. The parameter values used are $\alpha = 1$, $\Omega = 1/2$, $m = 1$, $\beta = -1.2345$. The solutions are attracted to the fixed point at the origin.

The equality in (44) and (45) is valid only on the fixed point. Hence the fixed point is a global attractor for the whole region $H \geq 0$ (see Fig. 1). Numerical simulations indicate that the same is also true for $\omega < 0$, as shown in Fig. 2.

For other parameter values numerical solutions point for an asymptotic diverging solution (see Fig. 3). For simplicity, let us consider the case $\omega = 0$ and the ansatz

$$H = \delta\psi_1, \tag{46}$$

with δ a constant. Hence

$$\dot{H} = \delta\dot{\psi}_1. \tag{47}$$

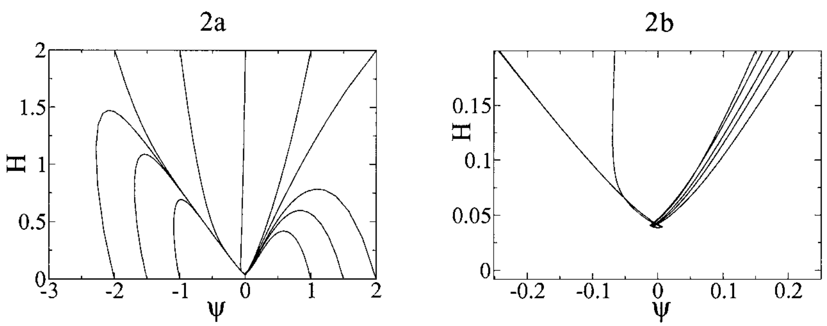

Fig. 2. Example of solutions projected in the (H, ψ_1) plane for $\omega = -0.5$. The other parameter values are the same as in Fig. 1. The solutions are attracted to the fixed point at $\psi_1 = 0$ and $H = \sqrt{-3\omega/\kappa}$.

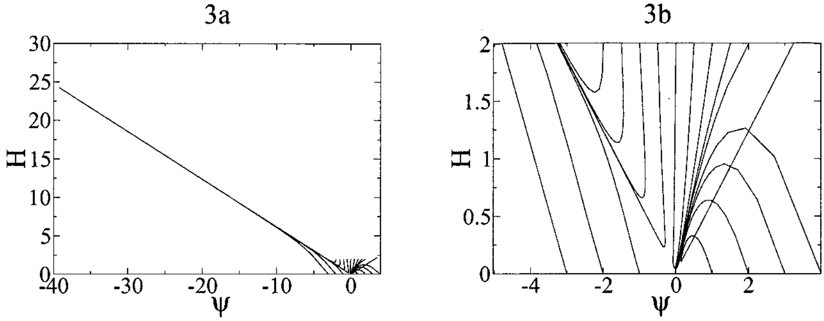

Fig. 3. Solution with $\omega = -0.3$. The remaining parameters are as in Fig. 1. Some solutions are diverging while others are still attracted to the fixed point at the origin. The quantities H, ψ_1, and ψ_2 diverge in a finite time.

From Eqs. (16), (17), and (47) we obtain

$$\delta\dot{\psi}_1 = \frac{1}{6}\psi_1^2\big(-12\delta^2 + m_1^2\kappa\big), \tag{48}$$

which is a closed equation for ψ_1, with solution

$$\psi_1(t) = \frac{6\delta\psi_1(0)}{6\delta + \big(12\delta^2 - \kappa m_1^2\big)\psi_1(0)t}, \tag{49}$$

where $\psi_1(0)$ is the initial condition for ψ_1 at $t = 0$. Using Eq. (12) and solution (49) easily yields $\psi_2(t)$:

$$\psi_2(t) = \pm\sqrt{-\frac{2}{\beta}}\,\frac{\sqrt{36\delta^4 - 12\delta^2 m_1^2\kappa - 18\Omega\delta^2 + m_1^4\kappa^2}\;\psi_1(0)}{\big(12\delta^2 - \kappa m_1^2\big)\psi_1(0)t + 6\delta}. \tag{50}$$

With the expressions for $\psi_1(t)$ and $\psi_2(t)$ it is straightforward (but a little bit cumbersome) to obtain the proportionality constant δ from the remaining equation (17):

$$\delta = \pm\frac{1}{6}\sqrt{6\kappa m_1^2 - 9\beta + 3\sqrt{-12\beta\kappa m_1^2 + 9\beta^2}}. \tag{51}$$

This value for δ agrees with the results of numerical integrations. It is important to note at this point that not necessarily every solution has the asymptotic behavior of Eq. (46). From numerical investigations it seems to exist a threshold in the initial values of ψ_2 and $\dot{\psi}_2$ such that above it the solutions converge asymptotically to the behavior obtained above. Below this threshold, solutions near the origin in the (H, ψ_1) plane are attracted toward it, exhibiting a similar behavior to the case of a single massive scalar field.

5. CONCLUDING REMARKS

The introduction of an extra massless field conserve some essential features of the dynamics of the one-field model of Gunzig *et a.* (2000, 2001), for some parameter values, particularly the spiraling solution near the fixed point at the origin. Nevertheless some new type of solutions exist that have no analog in the one-field case. For two self-interacting coupled fields the behavior is even more richer and a more exhaustive study is under preparation. Nevertheless, for some parameter ranges, new dynamical effects are obtained which modify the model in a substantial way. A discussion of which parameter values are of physical relevance is important for the future development of the present approach.

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